INVESTIGATION OF THE STABILITY OF SOME EXPLICIT DIFFERENCE SCHEMES IN THE INTEGRATION OF SAINT-VENANT EQUATIONS*

V.I. Koren'

Computations of the unsteady motion of water in rivers are based on the integration of a system of Saint-Venant equations:

$$\left.\begin{aligned} &\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+g\frac{\partial H}{\partial x}+\operatorname{sign}(v)\,\frac{gv^2}{K^2}-gi_0=0\\ &\frac{\partial \omega}{\partial t}+\omega\frac{\partial v}{\partial x}+v\frac{\partial \omega}{\partial x}=0 \end{aligned}\right\} \tag{1}$$

where v is flow velocity, averaged over the cross section; H is the depth of water; K is the discharge rate per unit area; i_0 is the slope of the bottom; ω is the cross sectional area; and g is the acceleration of gravity. Note that this system corresponds to the case when there is no lateral inflow along the river.

Many investigators have demonstrated that the magnitude of the inertial terms $(\frac{\partial v}{\partial t},\ v\frac{\partial v}{\partial x})$ in the equation of motion [first equation of system (1)] are substantially smaller, as a rule, than any of the other terms of this equation and, therefore, they are frequently omitted and a simpler system is used:

$$\left.\begin{aligned} &Q=\operatorname{sign}\left(i_0-\frac{\partial H}{\partial x}\right)\omega K\sqrt{\left|i_0-\frac{\partial H}{\partial x}\right|}\\ &\frac{\partial \omega}{\partial t}+\frac{\partial Q}{\partial x}=0 \end{aligned}\right\} \tag{2}$$

Systems (1) and (2) have no simple analytical solutions and, therefore, numerical methods are used for their integration. To provide for computational stability of the solutions of the difference equations we must satisfy some relationships between the steps of the difference grid (we shall examine only explicit schemes here). These relationships are determined mainly by the type of differential equations and also by the type of difference scheme selected.

System (1) belongs to the type of hyperbolic partial differential equations. We know from [3] that to provide for computational stability in the numerical integration of such equations by means of explicit schemes it is necessary to satisfy the Courant-Friedrichs-Lewy condition:

$$\frac{c\,\Delta t}{\Delta x}\leqslant 1, \tag{3}$$

here c is the rate of propagation of small perturbations in a moving stream [for system (1) $c = v \pm \sqrt{gH}$].

Let us demonstrate that this condition is necessary but not sufficient in the numerical integration of system (1). Assuming the channel to be prismatic and assuming an exponential

---

*Meteorology and Hydrology (Meteorologiya i gidrologiya), No. 1, 1967, pp. 42-48.

law of variation of K with depth ($K = \frac{1}{n} H^{\frac{x}{2}}$), we linearize system (1) with respect to steady flow with the characteristics H_0 and u_0:

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} + u_0 \frac{\partial u}{\partial x} + g \frac{\partial h}{\partial x} + 2 G_0 u - \varkappa \theta_0 h = 0 \\ &\frac{\partial h}{\partial t} + H_0 \frac{\partial u}{\partial x} + u_0 \frac{\partial h}{\partial x} = 0 \end{aligned}\right\} \tag{4}$$

Here we have introduced the notations $G_0 = \frac{gi_0}{u_0}$, $\theta_0 = \frac{gi_0}{H_0}$; u and h represent the deviations from u_0 and H_0 respectively.

Since (4) is a system of linear differential equations with constant coefficients that can be solved in the form of Fourier series, stability can be investigated according to the theory of J. von Neumann [4].

Let us examine two difference schemes that are extensively used in computations of unsteady motion: the Lax scheme

$$\frac{\partial f}{\partial t} = \frac{f_n^{m+1} - \frac{1}{2}\left(f_{n+1}^m + f_{n-1}^m\right)}{\Delta t}; \quad \frac{\partial f}{\partial x} = \frac{f_{n+1}^m = f_{n-1}^m}{2\Delta x} \tag{5}$$

and the central checkerboard scheme

$$\frac{\partial f}{\partial t} = \frac{f_n^{m+1} - f_n^{m-1}}{2\Delta t}; \quad \frac{\partial f}{\partial x} = \frac{f_{n+1}^m - f_{n-1}^m}{2\Delta x}, \tag{6}$$

where f is any differentiable function.

Having represented system (4) in difference form according to the Lax scheme, we find its conversion matrix [4]:

$$R(\Delta t, \alpha) \begin{vmatrix} (1-2 G_0 \Delta t) \cos\beta - \frac{u_0 \Delta t}{\Delta x} i \sin\beta & \varkappa\theta_0 \Delta t \cos\beta - \frac{g \Delta t}{\Delta x} i \sin\beta \\ -\frac{u_0 \Delta t}{\Delta x} i \sin\beta & \cos\beta - \frac{u_0 \Delta t}{\Delta x} i \sin\beta \end{vmatrix} \tag{7}$$

Here $\beta = \alpha\Delta x$, where α is a real wave number.

To provide for mathematical stability it is necessary that any eigenvalue of this matrix satisfy the von Neumann condition

$$|\lambda_i| \leqslant 1 + O(\Delta t) \tag{8}$$

for any value of the wave number. Let us determine the eigenvalues from the characteristic equation of the conversion matrix (7):

$$R(\Delta t, \alpha) - \lambda E| = 0, \tag{9}$$

where E is a unit matrix

Solving (9) for λ we obtain

$$\lambda_{1,2} = (1 - G_0 \Delta t) \cos\beta - \frac{u_0 \Delta t}{\Delta x} i \sin\beta \pm$$
$$\pm \sqrt{(G_0 \Delta t \cos\beta)^2 - \left(\frac{c_0 \Delta t}{\Delta x} \sin\beta\right)^2 - \frac{\varkappa}{2} \theta_0 \Delta t \frac{H_0 \Delta t}{\Delta x} i \sin 2\beta} \tag{10}$$

where $c_0 = \sqrt{gH_0}$. By letting Δt and Δx approach zero in such a way that $\Delta t/\Delta x$ = const and taking the absolute values of the eigenvalues, we can rewrite (10) as

$$|\lambda_{1,2}|^2 = 1 - \left[1 - \left(\frac{(u_0 \mp c_0)\Delta t}{\Delta x}\right)^2\right] \sin^2\beta + O(\Delta t).$$

Then, by finding a bound for the right-hand member of the equation (replacing the sine in it by unity), we obtain the stability condition in the sense of (8):

$$\frac{(u_0 \mp c_0)\Delta t}{\Delta x} < 1, \tag{12}$$

i.e., we have the Courant-Friedrichs-Lewy stability condition for hyperbolic equations.

But actually we run the risk that t may be insufficiently restricted by inequality (12), since it was obtained for the limiting case as Δt and Δx approach zero. However, for practical purposes we are interested not only in what will happen in the limit as $\Delta t \to 0$ and $\Delta x \to 0$, but also in what will happen for a point grid with finite Δt and Δx, which is used in numerical integration. Therefore, to obtain a criterion of computational stability let us require that the amplitudes of the solutions of the finite-difference equations not increase with time, i.e., that the inequality

$$|\lambda| < 1. \tag{13}$$

be satisfied.

Omitting in (10) the term containing θ_0 (since usually $\theta_0 \ll G_0$) and assuming that

$$\left(\frac{c_0\Delta t}{\Delta x}\sin\beta\right)^2 \gg \left(G_0\Delta t\cos\beta\right)^2,$$

we obtain the following expression for the absolute values of eigenvalues:

$$\begin{aligned} \left|\lambda_{1,2}\right|^2 \approx {} & \left[1-2G_0\Delta t \pm \gamma(G_0\Delta t)^2\right] - \\ & - \left\{\left[1-2G_0\Delta t \pm \gamma(G_0\Delta t)^2\right] - \right. \\ & \left. - \left[\frac{(u_0 \mp c_0)\Delta t}{\Delta x}\right]^2\right\}\sin^2\beta, \end{aligned} \tag{14}$$

where $\gamma = \frac{u_0}{c_0}$.

Equation (14) is approximate since it was assumed that

$$\begin{aligned} & \sqrt{\left(\frac{c_0\Delta t}{\Delta x}\sin\beta\right)^2 - \left(G_0\Delta t\cos\beta\right)^2} \approx \\ & \approx \frac{c_0\Delta t}{\Delta x}\sin\beta - \frac{(G_0\Delta t\cos\beta)^2}{2\frac{c_0\Delta t}{\Delta x}\sin\beta}, \end{aligned} \tag{15}$$

In what follows, we shall neglect the square of the second term in the right-hand member.

For inequality (13) to be fulfilled it is obviously necessary that, in addition to the Courant-Friedrichs-Lewy condition, the following inequality also be satisfied:

$$|\gamma|(G_0\Delta t)^2 + 2G_0\Delta t \leq 2. \tag{16}$$

By solving (16) for Δt we obtain

$$\Delta t \leq \frac{\sqrt{1+2|\gamma|}-1}{|\gamma|\,G_0}. \tag{17}$$

Without going through the detailed computations here, let us only note that when

$$\left(G_0\Delta t\cos\beta\right)^2 > \left(\frac{c_0\Delta t}{\Delta x}\sin\beta\right)^2$$

this restriction is not reinforced.

Thus, the maximum possible magnitude of the time step is determined by the magnitude of the resistance G_0 and the ratio of flow velocity to the rate of propagation of small perturbations

γ. It is obvious that in the absence of friction, restriction (17) is removed ($\Delta t < \infty$). If we do not allow for supercritical flows, the magnitude of $|\gamma|$ may vary within the limits $0 \leq |\gamma$ ≤ 1. Accordingly, λt also has two extreme values:

$$\left.\begin{aligned} \Delta t &\leqslant \frac{1}{G_0} && \text{for } \gamma = 0 \\ \Delta t &\leqslant \frac{\sqrt{3}-1}{G_0} && \text{for } |\gamma| = 1 \end{aligned}\right\} \qquad (18)$$

The first inequality in (18) corresponds to the stability criterion obtained in [1] for the simplified equation:

$$\frac{\partial u}{\partial t} + 2\,G_0 u = 0; \qquad (19)$$

thus, this criterion is a particular case of condition (17) when flow velocity is negligibly small compared to the rate of propagation of small perturbations ($\gamma \to 0$).

Consequently, in computations of unsteady motion in rivers according to this scheme it is necessary that, in addition to satisfying the Courant-Friedrichs-Lewy condition, we must take into account the size of the step of the difference grid according to inequality (17). The latter was obtained for a linearized system and, therefore, in the case of nonlinear equations (1) it can be carried over only as an approximation for a tentative estimate of the allowable step of the difference grid.

It should be noted that, generally, in computations of unsteady motion inequality (17) is automatically satisfied when condition (12) is fulfilled. But for small mountain rivers, when we must select a small distance step, Δt, determined from inequality (17), may prove to be smaller than required by the Courant-Friedrichs-Lewy condition.

Let us examine the possibility of removing restriction (17). To this end, let us approximate flow velocity in the friction term by the following relationship:

$$u = \delta\, u_n^{m+1} + \frac{1}{2}\,(1-\delta)\,(u_{n+1}^{m} + u_{n-1}^{m}), \qquad (20)$$

where $0 \leq \delta \leq 1$.

Since the criterion of stability (17) is determined mainly by the friction term (equal to $2G_0u$ in the linearized form), let us consider model equation (19). Having substituted the derivative in it according to the Lax scheme and using relationship (20), we find the conversion matrix (in this case it will consist of a single element):

$$R(\Delta t, \alpha) = \left| \frac{1 - 2\,G_0\,\Delta t\,(1-\delta)}{1 + 2\,G_0\,\Delta t\,\delta} \cos\beta \right|. \qquad (21)$$

Its eigenvalue is

$$\lambda = \left[1 - \frac{2\,G_0\,\Delta t}{1 + 2\,G_0\,\Delta t\,\delta}\right] \cos\beta. \qquad (22)$$

From (22) it follows that for $\delta \geq 1/2$ the quantity λ lies inside a unit circle or at its boundary for any time step of the difference grid, i.e., the restriction (17) is removed. If, however, $0 \leq \delta \leq 1/2$, to provide for stability we must satisfy the inequality

$$\Delta t \leqslant \frac{1}{G_0\,(1-2\delta)}, \qquad (23)$$

which for $\delta = 0$ corresponds to the stability criterion (18), examined earlier (for $\gamma = 0$).

By analogy with the Lax scheme, let us investigate the stability of the central checkerboard scheme. We limit ourselves to the examination of the simplified equation (19), since, as demonstrated above, the additional condition of stability, obtained for the complete system, practically coincides with the analogous condition corresponding to the model equation of motion. Let us represent equation (19) in difference form according to the central checkerboard scheme:

$$u_n^{m+1} = u_n^{m-1} - 4\,G_0\,\Delta t\,\delta u_n^{m+1} - 4\,G_0\,\Delta t\,(1-\delta)\,u_n^{m-1}. \qquad (24)$$

To reduce this three-layer problem to a two-layer problem, let us introduce a new dependent variable [4]: $w^{m+1} = u^m$. Then we obtain the following system:

$$\left.\begin{aligned} u^{m+1}_n &= \left[1 - \frac{4\,G_0\,\Delta t}{1 + 4\,G_0\,\Delta t\,\delta}\right] w^m_n \\ w^{m+1}_n &= u^m_n \end{aligned}\right\} \tag{25}$$

Its conversion matrix will be

$$R\,(\Delta t, \alpha) = \begin{vmatrix} 0 & 1 - \dfrac{4\,G_0\,\Delta t}{1 + 4\,G_0\,\Delta t\,\delta} \\ 1 & 0 \end{vmatrix}. \tag{26}$$

Let us find the eigenvalues

$$\lambda_{1,2} = \pm\sqrt{1 - \frac{4\,G_0\,\Delta t}{1 + 4\,G_0\Delta t\delta}}. \tag{27}$$

From (27) it follows that, by analogy with the Lax scheme, when $\delta \geq 1/2$ stability is ensured for any Δt. If, however, $\delta < 1/2$, it is necessary to satisfy inequality

$$\Delta t \leq \frac{1}{2\,G_0\,(1 - 2\,\delta)}, \tag{28}$$

i.e., for the central checkerboard scheme the bound on the time step of the difference grid is cut in half as compared with the Lax scheme.

It should be noted that in approximating the friction term $(G_0 u)$ by expressions $G_0 u = 1/2\ G_0\ (u^m_{n+1} + u^m_{n-1})$ and $G_0 u = G_0 u^m_n$ there is the risk that the central checkerboard scheme will be unstable for any $\Delta t \neq 0$, since the eigenvalues of the conversion matrix in this case are respectively equal to

$$\lambda_{1,2} = -\,G_0\,\Delta t \cos\beta \pm \sqrt{(G_0\,\Delta t \cos\beta)^2 + 1} \tag{29}$$

and

$$\lambda_{1,2} = -\,2\,G_0\,\Delta t \pm \sqrt{(2\,G_0\,\Delta t)^2 + 1}.$$

From (29) it follows that in both cases one of the eigenvalues may not lie inside the unit circle for all $\Delta t \neq 0$.

Let us now pass to the investigation of the stability of the simplified system on Saint-Venant equations (2). We linearize it for the case of a prismatic channel:

$$\left.\begin{aligned} q &= \left(\frac{\varkappa}{2} + \xi\right)\frac{Q_0}{H_0}\,h - \frac{Q_0}{2\,i_0}\,\frac{\partial h}{\partial x} \\ \frac{\partial h}{\partial t} &+ \frac{1}{B}\,\frac{\partial q}{\partial x} = 0 \end{aligned}\right\} \tag{30}$$

Here h and q are the deviations from the corresponding characteristics H_0 and Q_0 for steady motion; B is the width of the channel; $\varkappa$ is the hydraulic index of the channel; and ξ is the exponent in the relationship $\omega = f(H)$.

System (30) can be reduced to a single equation:

$$\frac{\partial f}{\partial t} - \sigma\frac{\partial^2 f}{\partial x^2} + a\frac{\partial f}{\partial x} = 0, \tag{31}$$

$$\sigma = \frac{Q_0}{2\,B i_0},\quad a = \left(\frac{\varkappa}{2} + \xi\right)u_0.$$

and f is the desired variable (h, q).

Equation (31) is parabolic with respect to the highest derivative with diffusion coefficient σ. The term $a\frac{\partial f}{\partial x}$ introduces wave properties and when $a \gg \sigma$ the equation becomes hyperbolic. Thus, system (30) is mixed in the general case. The result is that the conditions of computational stability, corresponding both to parabolic and hyperbolic equations, cannot be extended directly to the system in question.

Let us find the stability criterion of system (30) for its numerical integration according to the explicit scheme considered in [2]:

$$\frac{\partial h}{\partial t} = \frac{h_n^{m+1} - h_n^m}{\Delta t}, \quad \frac{\partial h}{\partial x} = \frac{h_{n+1}^m - h_n^m}{\Delta x},$$

$$\frac{\partial q}{\partial x} = \frac{q_{n+\frac{1}{2}}^m - q_{n-\frac{1}{2}}^m}{\Delta x}. \tag{32}$$

Having substituted (32) in (30) and having reduced the system of difference equations to a single equation in h, we obtain

$$h_n^{m+1} = h_n^m - \frac{a\Delta t}{2\Delta x}\left(h_{n+1}^m - h_{n-1}^m\right) + \frac{\sigma\Delta t}{\Delta x^2}\left(h_{n+1}^m + h_{n-1}^m\right) - \frac{2\sigma\Delta t}{\Delta x^2}h_n^m. \tag{33}$$

Its conversion matrix will be

$$R(\Delta t, a) = \left| 1 - \left(\frac{4\sigma\Delta t}{\Delta x^2}\sin^2\frac{\beta}{2} + \frac{a\Delta t}{\Delta x}i\sin\beta\right|\right. \tag{34}$$

From (34) it follows that to satisfy the von Neumann condition [4] we must satisfy the inequality

$$\frac{\sigma\Delta t}{\Delta x^2} \leqslant \frac{1}{2}, \tag{35}$$

i.e., we obtain the stability criterion characteristic for parabolic equations. If we require that inequality (13) be satisfied, i.e., allow for the fact that $\Delta t \neq 0$, the stability criterion will be

$$\left(\frac{2\sigma\Delta t}{\Delta x^2}\right)^2 + \left(\frac{a\Delta t}{2\Delta x}\right)^2 < 1, \tag{36}$$

i.e.,

$$\Delta t \leqslant \frac{\Delta x^2}{2\sigma}\sqrt{\frac{1}{1 + \left(\frac{a\Delta x}{4\sigma}\right)^2}}. \tag{37}$$

For rivers, $\sigma \gg a$, as a rule, and, therefore, the role of the wave component in (37) is small.

For small distance steps, condition (37) is much more stringent than the Courant-Friedrichs-Lewy criterion for the complete system (1). As Δx increases, this relationship changes rapidly and the stability criterion (37) becomes less stringent than (12).

To determine the possibility of extending the stability criteria obtained to nonlinear equations, numerical experiments were performed for a broad range of variation of initial and boundary conditions. Computations from the complete system (1) and from the simplified system (2) showed that conditions (17), (28), and (37) also remain valid for nonlinear equations.

REFERENCES

1. Vasil'yev, O. F., et al. Numerical Method of Computing the Propagation of Long Waves in Open Channels and its Application to the Flood Problem. Proc. Academy of Sciences of the USSR (Doklady AN SSSR), Vol. 151, No. 3, 1963.

2. Kalinin, G. P. and L. S. Kuchment. Numerical Methods of Solving the Saint-Venant Equations for Computation of Unsteady Water Motion in Rivers. Meteorology and Hydrology (Meteorologiya i gidrologiya), No. 6, 1963.
3. Courant, R., K. Friedrichs, and H. Lewy. Difference Equations of Mathematical Physics. Progress in Mathematical Sciences (UMN), No. 8, 1941.
4. Richtmyer, R. D. Difference Methods for Initial-Value Problems. Foreign Literature Press, Moscow, 1960.

Hydrometeorological Scientific Research Center of the USSR

Received March 31, 1966

Soviet Hydrology: Selected Papers

No. 1, 1967

METHOD OF FORECASTING ICE PHENOMENA IN RIVERS*

N. A. Bagrov and A. P. Kukhto

Ice phenomena in rivers are closely associated with seasonal variations in heat exchange in the atmosphere-water-ice system and with some hydrological factors. However, the dominating role in all ice phenomena is undoubtedly played by the thermal process.

The characteristics of atmospheric circulation over a sufficiently vast area must be used as initial data in long-range forecasts of ice phenoma. In our opinion, sufficient account is not always taken of these characteristics.

An attempt to forecast ice phenomena on the basis of a more detailed, quantitative estimate of initial atmospheric processes is made here without allowance for the effect of hydrologic factors in the first approximation.

The idea of using some objective circulation criteria for hydrological forecasts is not new. There are some studies where various "circulation indexes" are used as predictors [3].

It is obvious, however, that atmospheric circulation over a vast area cannot be characterized by a single quantity (be it the meridional index, the zonal index, or both). Therefore, for an objective estimate of atmospheric circulation characteristics over a sufficiently vast area, we believe it advisable to use a set of coefficients of the expansion of a meteorological field, in particular the H_{500} (500-mb) field, in Chebyshev polynomials.

We know that the H_{500} field adequately describes the prevailing flow in the troposphere and is used effectively by synoptic meteorologists in weather forecasting. Furthermore, relatively complete absolute 500-mb topography charts are available for a sufficiently long series of years.

The general theory and technique of expansion and meteorological fields in Chebyshev polynomials are described in [1].

The A_{ij} coefficients represent some generalized circulation indexes or parameters. A set of these indexes can describe with some degree of accuracy the circulation on any synoptic chart and it can do it the more accurately, the more complete is the set of indexes taken into account.

For initial data we used the periodic charts of average H_{500} values, as it was done in [4]. The data were taken at 117 points on a "trapezoid" from 35° to 75°N (9 points) and from 30°W to 90°E (13 points). As a result, 47 coefficients A_{ij} ($i = 0, 1, \ldots, 5$; $j = 0, 1, \ldots, 6$) were obtained for each chart.

The main idea of an analytical representation of a meteorological field is to substitute the entire initial information, taken in a system of discrete points, by relatively few expansion coefficients, the number of which is several times (4-10) smaller than the number of points.

*Meteorology and Hydrology (Meteorologiya i Gidrologiya), No. 2, 1967, pp. 22-28.